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THE REAL FUNCTION DEFINED BY $X^y = Y^x$.

By E. J. MOULTON, Northwestern University.

It is obvious that for powers the commutative law does not hold in general. That is,

$$a^b \neq b^a, \text{ in general.}$$

But in some cases the letters may be interchanged, as for example when $a = b$, or when $a = 2$, $b = 4$. In looking for other values of the letters for which they may be interchanged, I have made the following discussion of the complete real locus of the equation

$$x^y = y^x. \tag{1}$$

I. THE FIRST QUADRANT.

First, let us consider only positive values of x and y. When an exponent is fractional, as in $x^{1/2}$, or irrational, we shall understand by the symbol the positive real value of the indicated power. Then our equation is equivalent to

$$y \log x = x \log y. \tag{2}$$

To discuss this equation we consider the equation

$$\frac{\log x}{x} = \frac{\log y}{y} = k. \tag{3}$$

The parameter k is real. Now let us plot the equations

$$u = \log t \quad \text{and} \quad u = kt. \tag{4}$$

Suppose these curves intersect at $Q_1(t_1, u_1)$ and $Q_2(t_2, u_2)$. Then we have

$$\frac{\log t_1}{t_1} = \frac{\log t_2}{t_2} = k.$$

Hence we obtain the following four solutions of (3): $x = t_1, y = t_1$; $x = t_1, y = t_2$; $x = t_2, y = t_1$; $x = t_2, y = t_2$; or in a brief notation,

$$P_{11}(t_1, t_1),\ P_{12}(t_1, t_2),\ P_{21}(t_2, t_1),\ P_{22}(t_2, t_2).$$

Thus, by finding the intersections of equations (4) we find solutions of (3). It is also clear that the values of x and y in a solution of (3) are values of t at points of intersection of the curves (4). Hence, we shall find all the solutions of (3) or of (1) by the method indicated.

When $k = 0$ the curves (4) have only one point of intersection Q_1 (see Fig. 1). Hence, corresponding to each value of $k = 0$ there is just one point P_{11} on (1).

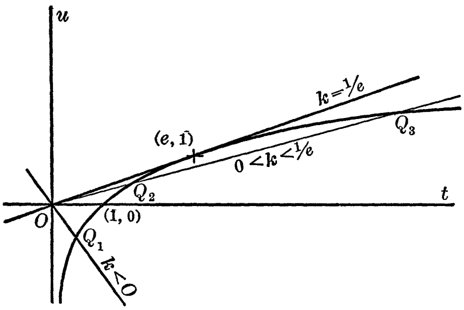

FIG. 1. $u = \log t$ and $u = kt$.

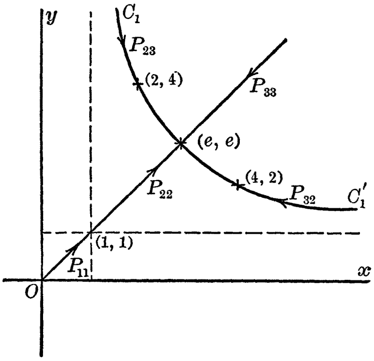

FIG. 2. Part of locus: $x^y = y^x$.

As the parameter k varies from $-\infty$ to 0, t_1 varies from 0 to 1, and hence P_{11} moves along the line $y = x$ from (0, 0) to (1, 1), the former point being excluded, the latter included.

When $0 < k < 1/e$, $e = 2.71828\cdots$, the curves (4) have two points of intersection, Q_2 and Q_3. Hence, corresponding to each of these values of k there are four points, P_{22}, P_{23}, P_{32}, P_{33}, on (1). As the parameter k varies from 0 to 1, t_2 varies from 1 to e and t_3 from ∞ to e; and hence P_{22} moves along $y = x$ from (1, 1) to (e, e), P_{23} comes from infinity along a curve C_1 asymptotic to $x = 1$, and approaches (e, e), P_{32} comes from infinity along a curve C_1' asymptotic to $y = 1$, and approaches (e, e), and P_{23} comes from infinity along $y = x$ to (e, e). In describing C_1 the point P_{23} moves monotonically to the right and down; in describing C_1' the point P_{32} moves monotonically to the left and up. The point (2, 4) is found to lie on C_1 and (4, 2) on C_1'. See Fig. 2.

When $k = 1/e$ the curves (4) have one point of intersection $(e, 1)$. The corresponding point of (1) is (e, e).

When $k > e$ the curves (4) do not intersect and there is no corresponding point of (1).

II. THE GENERAL POSITION.

If either x or y is negative and the other fractional or irrational, it is not eas to describe simply the unique meanings to be attached in general to each of th expressions x^y and y^x. For example, $(-2)^{3/7}$ is seven-valued, and $(-2)^e$ is infinitely-many-valued, each value being a complex number. Accordingly, we shall say that (a, b) is a point of the locus (1) if by proper choice from the values of a^b and b^a we have $a^b = b^a$.

Obviously the equation is satisfied if $y = x$, except at $(0, 0)$. If $x = 0, y = 0$, we have the symbol 0^0, which is customarily considered meaningless (not many-valued), except as it is associated with the limit of a function. If we take $y = x$ in (1) and let $x = 0$ we obtain the symbolic equation $0^0 = 0^0$, and in this sense the point $(0, 0)$ is on the locus of (1) also.

Equation (1) is equivalent to

$$e^{y \log x} = e^{x \log y}.$$

The logarithmic function is many-valued; this is shown explicitly by writing our equation

$$e^{y[\mathrm{Log}\,|x| + i(\theta + 2p\pi)]} = e^{x[\mathrm{Log}\,|y| + i(\phi + 2q\pi)]}.$$

Here $\mathrm{Log}\,|x|$ is the single-valued real logarithm of the absolute value of x, and $\mathrm{Log}\,|y|$ of y; p and q are positive or negative integers or zero, and θ and ϕ are zero or π according as x and y, respectively, are positive or negative numbers. These two exponentials are equal if and only if the exponents differ by $2n\pi i$, n an integer. Hence (1) is equivalent to

$$y\,\mathrm{Log}\,|x| + i(\theta + 2p\pi)y = x\,\mathrm{Log}\,|y| + i(\phi + 2q\pi)x + 2n\pi i.$$

For real solutions we must have simultaneously

$$y\,\mathrm{Log}\,|x| = x\,\mathrm{Log}\,|y|, \tag{5a}$$

$$(\theta + 2p\pi)y = (\phi + 2q\pi)x + 2n\pi. \tag{5b}$$

The graph of (5a) is obtained by the method used in discussing (2). The figures corresponding to Fig. 1 and Fig. 2 are given in Fig. 3 and Fig. 4. The

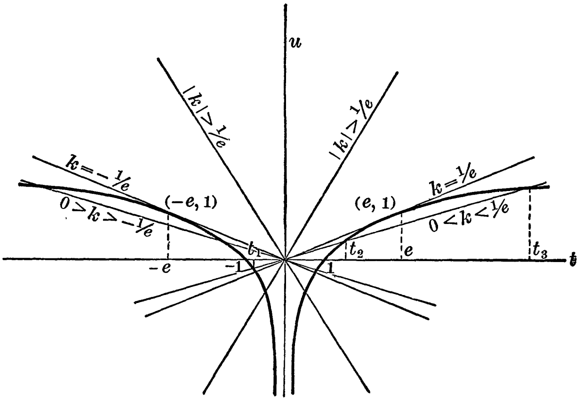

FIG. 3. $u = \log|t|$ and $u = kt$.

points of the locus in Fig. 4 which satisfy (5b) for some choice of p, q and n constitute the graph of (1). We have already seen that the line $y = x$ and the curve C_1 are parts of the locus; it remains to consider C_2, C_3 and C_4.

In the second quadrant $\theta = \pi$, $\phi = 0$, and (3b) becomes

$$(2p + 1)y = 2qx + 2n. \tag{7}$$

Not every point of C_2 *is on* $x^y = y^x$, for p, q, n cannot be chosen so that (7) is satisfied by $(-1, 1)$, since otherwise we would have an odd number equal to an even number. *But the points of* C_2 *on* $x^y = y^x$ *in the vicinity of any point of* C_2 *form a dense point aggregate.*[1] To prove this we let (a, b) be any point on C_2 and show that it is a limit point of a set of intersections of (7) with C_2. We consider only the lines for which $q = 0$,

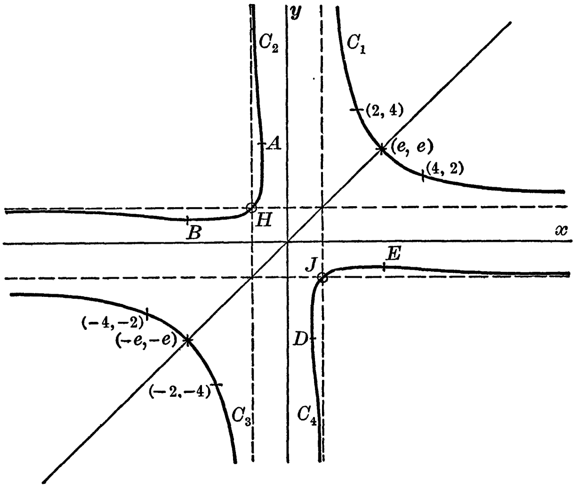

FIG. 4. Locus $y \log | x | = x \log | y |$. Includes locus $x^y = y^x$.

Special points: $A(-.7569\cdots, e)$; $B(-e, .7569\cdots)$; $D(.7569\cdots, -e)$; $E(e, -.7569\cdots)$; $H(-1, 1)$; $J(1, -1)$. B is a minimum and E a maximum of the locus.

$$y = \frac{2n}{2p + 1},$$

which are parallel to the x-axis. It is then clearly sufficient to show that b is a lower limit point of a set of numbers of the form $2n/(2p + 1)$. Such a set is formed by taking in succession $p = 1, 2, 3, \cdots$, and choosing n so that

$$b(2p + 1) < 2n \leqq b(2p + 1) + 2. \tag{8}$$

Since the extremes of these inequalities differ by 2, there is just one integer $2n$ satisfying (8). And since (8) may be written

$$0 < \frac{2n}{2p + 1} - b \leqq \frac{2}{2p + 1}$$

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[1] PIERPONT, Theory of Functions of a Real Variable, I, p. 167.

it is seen that in this set are an infinity of numbers greater than b and that they approach b as a limit as p increases indefinitely.

In the fourth quadrant the situation is precisely similar to that in the second.

In the third quadrant $\theta = \pi$, $\phi = \pi$, and (5b) becomes

$$(2p + 1)y = (2q + 1)x + 2n. \tag{9}$$

Hence the points of C_3 on $x^y = y^x$ in the vicinity of any point of C_3 form a dense point aggregate. For in this case we need consider only the lines when $q = p$,

$$y = x + 2n/(2p + 1),$$

which are parallel and have intercepts which are dense on the y-axis. Since C_3 is nowhere parallel to these lines the conclusion is obvious.

The points $(-2, -4)$ and $(-4, -2)$, in particular, lie on $x^y = y^x$.

Conclusion.—Fig. 4 is the graph of $\frac{\operatorname{Log}|x|}{x} = \frac{\operatorname{Log}|y|}{y}$. It contains all points of the graph of $x^y = y^x$; also it contains other points. Every point on the line $y = x$ and every point on the curve C_1 lies on $x^y = y^x$; the points H and J are not on the locus, but there is an everywhere dense set of points on C_1, C_2, C_3 lying on the locus. I do not know whether there are points on C_3 which are not on $x^y = y^x$, or how many points on C_2 and C_4 are not on $x^y = y^x$.

CONCERNING ROULETTES.

By GOLDIE HORTON, University of Texas.

I. If one curve rolls on another the curve traced by any point in the plane of the rolling curve is called a roulette. The rolling curve is called the moving centrode M and the fixed curve is called the fixed centrode F. To safely apply the methods of infinitesimals we shall suppose that the functions employed in defining the moving and fixed centrodes have continuous first and second derivatives.

(1) As M rolls on F, a line l in the plane of M and a point P on l go into a line l' and a point P' on l', and the point of contact T of M and F moves to T'. The range P on l is congruent to the range P' on l' and hence the pencil PT is projective with the pencil $P'T'$. Since the limiting position of the intersection of PT and $P'T'$ as P' moves back to P is the center of curvature at P of the roulette traced by P, we have

THEOREM 1. *The locus of the centers of curvature of the elements described simultaneously by all the points of a line in the plane of the moving centrode, for an infinitesimal movement, is a conic tangent to the moving centrode and also to the fixed centrode at the instantaneous center of rotation.*

Bresse attributes this theorem to Rivals.[1] It was Mannheim, however, who

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[1] See *Journal de l'École Polytechnique*, Cahier 35, p. 112.